Interacting with mathematics in Key Stage 3

Year 9 geometrical reasoning: mini-pack

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Year 9 geometrical reasoning: sample unit

Introduction

The Royal Society/Joint Mathematical Council report *Teaching and learning geometry 11–19* highlighted the need to reinforce the position of geometry in the curriculum (Royal Society, 2001; www.royalsoc.ac.uk). The 2000 revision of the National Curriculum also gives more emphasis to geometry in Key Stages 3 and 4. This unit aims to make a significant contribution to the development of teaching and learning of geometrical reasoning. It is planned for the end of Year 9 in order to provide an opportunity to draw together pupils' existing geometrical knowledge within a more rigorous framework and to make a bridge into Key Stage 4. As developments in schools evolve over time, it is expected that some of the content should move to earlier in Year 9, perhaps to the autumn term, or even into Year 8.

This unit has been developed through flexible use of the *Sample medium-term plans for mathematics*.

In planning the unit several decisions were made that affect the medium-term plans for mathematics.

- The objectives for the unit are mainly drawn from Shape, space and measures 4.
- The objective 'Use and interpret maps and scale drawings' will need to be covered in Shape, space and measures 3.
- The objective 'Calculate the surface area and volumes of right prisms' will need to be covered in Shape, space and measures 2.

The emphasis of the unit is on reasoning rather than content, much of which may be familiar to pupils. The approach allows greater rigour to be developed by re-establishing familiar definitions and properties into a logical hierarchy. Phase 1 develops this hierarchy using images and language to model the presentation of a chain of reasoning both orally and in written form. These strategies are transferred to solving geometrical problems in phase 2.

Phase 1 (three or four lessons)

- This first phase uses visualisation and reconstruction exercises as oral and mental starters. In these, pupils form and manipulate images of lines and shapes, developing a feel for spatial relationships, an awareness of the generality of certain results and the language needed to describe what they see.
- The phase uses 'build-ups' (simple line diagrams presented on acetate overlays or as tracings) in the main part of the lesson to demonstrate how certain 'given' facts about lines and angles can be combined to establish new facts, including reviewing the process and recording steps in a logical sequence.
- Keeping the emphasis on relationships between geometrical results and their ordering helps pupils appreciate deductive argument and why they are asked to explain what may seem obvious to them.
- The phase seeks to clarify the distinction between demonstration of particular cases and an argument that must hold for all cases.

Phase 2 (about three lessons)

- This phase applies properties established in phase 1 to the solution of problems that involve constructing geometrical diagrams and analysing how they are built up.
- It develops written solutions, where the ‘given’ facts (assumptions) are stated as justification in logically ordered explanations and proofs.
- The phase reviews established facts and properties and the connections between them, so that pupils begin to gain a sense of a logical hierarchy.

Objectives

A Distinguish between conventions, definitions and derived properties; *distinguish between practical demonstration and proof; know underlying assumptions, recognising their importance and limitations, and the effect of varying them.*

B Explain how to find, calculate and use:

- the sums of the interior and exterior angles of quadrilaterals, pentagons and hexagons;
- the interior and exterior angles of regular polygons.

C **Solve problems using properties of angles, of parallel and intersecting lines, and of triangles and other polygons,** justifying inferences and explaining reasoning with diagrams and text.

D Visualise and use 2-D representations of 3-D objects; analyse 3-D shapes through 2-D projections, including plans and elevations.

E Transform 2-D shapes by combinations of translations and rotations; **know that translations and rotations preserve length and angle and map objects on to congruent images.**

F **Present a concise, reasoned argument, using symbols, diagrams and related explanatory text;** *give reasons for choice of presentation, explaining selected features and showing insight into the problem’s structure.*

G Suggest extensions to problems, conjecture and generalise; identify exceptional cases or counter-examples, explaining why; *justify generalisations, arguments or solutions; pose extra constraints and investigate whether particular cases can be generalised further.*

Differentiation

The dynamic approach using visualisations and the scaffolding to thinking provided by the ‘build-ups’ creates opportunities for a wide range of pupils to participate. In each case the level of challenge and autonomy can be varied for groups working at different levels. For example:

- Visualisations
  - partly completed images could be drawn to provide a starting point which pupils then animate in their imaginations;
  - the number of steps in each visualisation can be reduced;
  - some pupils could be allowed to support their verbal descriptions with small sketches.

- Build-ups
  - whole-class demonstration could guide pupils through individual marking of tracing-paper build-ups;
  - each build-up is reconstructed following a demonstration;
  - the emphasis is on accurate oral reasoning supported by teacher modelling of written argument.

Extensions and harder problems are included in phase 2. More explicit differentiation strategies are:

- varying the degree of difficulty in accessing problems according to use of diagrams and text:
  - the diagram is presented with the problem;
  - a diagram has to be constructed from precise instructions;
  - a diagram is required but the text does not give explicit instructions.
- varying requirements for explanations:
  - clear oral explanation but abbreviated written explanation;
  - written explanation supported with part of the text given (e.g. statements provided but reasons to be added);
  - full written explanation required.

Resources

- Build-ups (resource sheets):
  - as acetate overlays for OHP
  - as tracing-paper copies for pupils
- Tracing paper (good quality)
- CD (For visualisations, tracks 1–7, reading from *Alice's Adventures in Wonderland*, track 8)
- Whiteboards, straws, thin card
- Copies of selected geometry problems
- Start of a flowchart of logical reasoning (poster, OHT or handout)
- Supplementary notes (pages 10–36 in this booklet)
  - Prompts for oral and mental starters in phase 1
  - Prompts for main activities in phase 1
  - Prompts for main activities in phase 2
  - Prompts for first plenary in phase 1
  - Prompts for final plenary in phase 2
  - Problem bank

Notes

The prompts feature scripted notes for both visualisation exercises and build-ups. As an alternative to playing the visualisation exercises on the CD, a prepared script helps you to control the language, sequence and pace of presentation and facilitates later discussion. Teachers trialling the idea of build-ups commented that the scripts for these gave them a good understanding of how to sequence and manage class discussion with the overlays. In quite a short time familiarity with the build-ups meant that they became more independent of the detailed notes.

Mathematics departments with good access to computers will recognise the potential for using dynamic geometry software and make appropriate adaptations to the unit. However, this is not essential and all departments should be able to adopt the approach outlined by making good use of the OHP, with the build-ups provided.

Key mathematical terms and notation

convention, definition, derived property (fact)

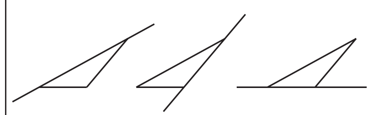
justify, deduce, generalise, prove, proof

vertically opposite, alternate, corresponding angles

interior, exterior, supplementary angles

plane, section/cross-section, plan, elevation, projection

Unit plan

Oral and mental starter	Main teaching	Notes	Plenary
Objectives E, F (Framework p. 185) **From words to pictures** Visualisation exercises (consider using some of the following exercises a week or two before starting this unit): 1 Two intersecting lines 2 Two parallels and a transversal 3 Extending the sides of a triangle 4 Extending the sides of a parallelogram 5 Three intersecting lines 6 Two pairs of parallel lines 7 Halving a rectangle (See 'Prompts for oral and mental starters in phase 1' and audio tracks 1–7.)	**Phase 1 (three or four lessons)** Objectives A, B, F, G (Framework pp. 179–183) **From pictures to words – establishing the foundations of proof** Discuss conventions and definitions. Use diagrams and commentary to model the process of deductive geometrical argument. Using acetate overlays or tracing paper, build up sequences 1 to 5 establishing definitions (corresponding and alternate angles, parallel lines) and proving geometric properties (vertically opposite angles are equal, pairs of alternate angles on parallel lines are equal). Discuss what is 'given' and how the new property is established or proved from givens. Model recording of the build-up jointly as a class. Gradually increase the level of formality. Pupils reconstruct build-ups rehearsing language and refining written form of recording proof. (See 'Prompts for main activities in phase 1'.)	A fourth lesson may be needed, depending on pupils' response to the build-ups. Conventions, definitions and facts (derived properties) are defined in the Framework p. 179. Using the resource sheet 'Start of a flowchart of logical reasoning' as a teacher prompt, gradually construct a poster size flow chart of related geometrical facts (see 'Prompts for final plenary in phase 2').	Present a short extract from the trial in *Alice's Adventures in Wonderland* (reading on CD track 8, text in 'Prompts for first plenary in phase 1'). Discuss false assumptions and unjustified conclusions. Draw analogy with 'givens' and 'derived properties' in geometrical argument.
	Proving the angle sum of a triangle Build up sequences 6 and 7, proving interior angles of a triangle add to 180º. Follow-up activities. Using the same given facts, pupils reconstruct the argument. • Can the build-ups be used to construct the argument for any type of triangle? Pupils devise figures which form different types of triangle (e.g. isosceles, scalene, obtuse angle, right angle) and reconstruct the proof as the triangle is formed. • Can the build-ups be created for one given triangle beginning from any side? Pupils visualise the stages of the build-up which would be generated by drawing a line parallel to any one side of the triangle (see diagrams) and reconstruct proof from the 'freeze frame' of the final picture.	The purpose of the follow-up activities is to illustrate the generality of the argument. Allocate types of triangles to different groups, with mini-plenaries to share findings. 	Will the argument apply to all triangles? Distinguish between proof and demonstration, e.g. for a selection of triangles, drawing and measuring the angles or cutting out, tearing off the corners and fitting them together.
	Proving the angle sum of a polygon Develop similar approaches to the sum of interior angles of polygons. Three start points all based on 'building up' the interior angles using triangles with adjoining sides the same length. Position a vertex from each triangle: • at a point within the polygon; • at a point on a side of the polygon; • at a vertex of the polygon (the usual approach). Encourage pupils to build up for themselves establishing levels of algebraic presentation appropriate to the group, relating final algebraic form back to context. 180*n* – 360 — *n* triangles subtract the full turn at the point within the polygon 180(*n* – 1) –180 — (*n* – 1) triangles subtract the $\frac{1}{2}$ turn at the point on the side of the polygon 180(*n* – 2) — (*n* – 2) triangles	Extension: For any of the three versions of a general form simplify and use to find the size of one interior angle of a regular polygon 180 – 360/*n*. Relate the algebraic form to the sum of exterior angles and equality of each exterior angle in a regular polygon.	Invite pupils to talk through their written explanations with the rest of the class. (See notes under phase 2 below.)

<table>
<tr><th>Oral and mental starter</th><th>Main teaching</th><th>Notes</th><th>Plenary</th></tr>
<tr>
<td>Objectives D, F
(Framework pp. 199–201)

3-D work, using sketches on paper or whiteboard. For example:
• Sketch and label a cube, identify parallel and perpendicular lines, compare diagonals, etc.
• Discuss possible orientations of two lines or planes in space (straws and card to aid visualisation).
• Describe solid shapes from plans and elevations.
• Explore sections obtained by slicing a cube.</td>
<td>Phase 2 (three lessons)
Objectives B, C, F, G (Framework pp. 17, 183, 185)

Solving problems
Select problems from the problem bank. Work through one or two examples together, setting out appropriate expectations in terms of written explanations. Model the task and support pupils' explanations by setting out the stages in the process:
• Clarify the task.
• Build up the chain of reasoning.
• Deduce some information.
• Conclude the argument.
Explain conventions for written explanations: recording line by line, giving reasons in brackets after each statement or equation, using symbols such as ∠, Δ, //.
Provide pupils with the following list of given facts to use as justification for their conclusions (all except the last two were developed in phase 1):
• Angles on a straight line add to 180°.
• Vertically opposite angles are equal.
• Pairs of corresponding angles on parallel lines are equal.
• Pairs of alternate angles on parallel lines are equal.
• The interior angles of a triangle add to 180°.
• The interior angles of an n-sided polygon add to $180(n - 2)°$.
• Angles opposite the equal sides of an isosceles triangle are equal.
• In a triangle with two equal angles, the sides opposite the equal angles are equal and the triangle is isosceles.
Organise pupils in pairs to:
• solve the problem on their own if they can;
• compare notes, explaining their reasoning;
• agree on a solution and write out the proof;
• present an alternative solution, if appropriate.</td>
<td>Use alternative wordings for the problems according to the level of challenge required.
Differentiation in written arguments:
• Pupils present a simplified explanation, omitting some points of detail.
• Pupils are given a partially completed explanation, to add reasons for each statement, or to put statements in logical order.
• Pupils prepare a full written explanation.</td>
<td>As appropriate:
• Ask selected pairs of pupils to prepare a written explanation to share with the class, preferably on an OHT for ease of display.
• As the pupils explain their steps, get the class to ask questions for clarification or adopt the role of 'doubters'.
• Discuss clarity of explanations and soundness of reasoning.
• Compare and evaluate different explanations.

Flowchart of logical reasoning
To review the unit, show 'Start of a flowchart of logical reasoning' (wall chart or OHT). Discuss the hierarchy of facts and links between them.</td>
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Prompts for oral and mental starters in phase 1

Visualisation exercises

These activities aim to support pupils in developing a structure to their reasoning as they work from words to pictures.

Visualisation exercises make excellent starters to lessons. They can be used to help pupils construct and control mind pictures, see geometrical relationships and develop the language to describe what they see. They need only be quite short, perhaps no more than 2 or 3 minutes. The idea is to ask the class to imagine a simple picture of lines or shapes, building up or altering the picture in various ways and asking them to notice features, what changes and what stays the same. This is usually followed by a whole-class discussion to reconstruct what they saw, probably taking quite a bit longer than the visualisation itself. Here are some principles for conducting successful visualisation exercises.

The visualisation

- Begin by asking the class to shut their eyes – you are going to ask them to imagine some pictures and they are not allowed to draw anything.
- Tell them they need to concentrate as you will not be repeating what you say.
- Speak clearly and deliberately, choosing your words carefully and pausing between each instruction.
- Judge the timing of each pause carefully, long enough for all pupils to develop their image, but not so long as to lose concentration.
- It is important to stress that pupils are not allowed to speak or ask questions during the visualisation – any questions you ask, pupils think to themselves what is the answer and do not respond aloud.

The reconstruction

- Once you have finished the visualisation exercise, invite pupils to describe what they saw, working through the sequence of images and drawing out particular points.
- Initially they should work descriptively, to develop more precise use of language and so that others have to work at seeing what they see. This may be an opportunity to reinforce or introduce new vocabulary.
- Be prepared for different and unexpected responses and do not treat them as wrong. You are asking pupils to imagine and language is rarely unambiguous. The most fruitful discussions can happen in such situations, for example about different ways in which a set of lines can be arranged.
- There is a discipline to controlling and working with mind pictures, so resist the temptation to draw pictures until later in the reconstruction, when clarity is needed but words fail!

Some further points

- Unless you are very experienced at conducting these exercises, it helps to prepare a script in advance. For this unit, scripts are given below and on the CD, tracks 1 to 7.
- Visualisation exercises provide a good opportunity to develop use of mathematical vocabulary, but choose your words carefully. If you use a word which may be unfamiliar (e.g. ‘transversal’), make sure that you explain it before starting the exercise, rather than breaking pupils’ concentration. To avoid being over-technical you may sometimes use everyday terms; for example, you might ask the class to ‘push’ lines rather than translate them.
- When you first start these exercises, pupils may not find them easy. Build up gradually, as everyone gets better with experience.
- Be prepared for some pupils to express concern that they cannot see anything! It is best not to put them under any pressure. Reassure them that this is quite normal and they might begin to see something as other pupils reconstruct their images.

Some visualisation scripts

Some examples of visualisation scripts follow. They are all in two dimensions. In several examples, you will need to establish that lines are intended to be infinite. Check that pupils are familiar with any key vocabulary that you intend to use (some alternative vocabulary is in brackets). It is presumed that each visualisation will be followed by a discussion, reconstructing what pupils saw. If pupils find the exercise difficult, particularly if it is a new experience, try pausing part-way through to discuss the first part before proceeding with the second.

The visualisations are:

1 Two intersecting lines

2 Two parallels and a transversal

3 Extending the sides of a triangle

4 Extending the sides of a parallelogram

5 Three intersecting lines

6 Two pairs of parallel lines

7 Halving a rectangle

The first two examples encourage pupils to develop their intuitions about lines and angles using a dynamic approach. The first example is initially concerned with angles on a straight line and vertically opposite angles. By way of explanation pupils might, for example, make arguments about angles increasing or decreasing at the same rate. This and the second example could be used to focus attention on corresponding and alternate angles.

Visualisation 1: Two intersecting lines

Imagine a line … Make it horizontal ... Imagine that it extends to infinity in both directions.

Imagine a vertical line, crossing (intersecting) the horizontal one and extending to infinity in both directions … Can you see four right angles?

Fix your mind on the point of intersection of the two lines … Keeping the horizontal line fixed, **very slowly** rotate the vertical line clockwise about this point.

As you slowly rotate the line, think to yourself: What happens to the right angles?

Continue rotating the line very slowly until it is at 45°, then stop … How many angles of 45° can you see? … What is the size of the other angles?

Now, keeping the sloping line fixed at 45°, **very slowly** move (translate) the horizontal line upwards, keeping it horizontal.

As you slowly move (translate) the line, what happens to the point of intersection? What happens to the angles at this point? … Do they change in size?

Now slowly move (translate) the horizontal line downwards, past its original position. What happens to the point of intersection? What happens to the angles at this point? … Do they change in size?

Visualisation 2: Two parallels and a transversal

Imagine two parallel lines of infinite length … Make them horizontal, not too far apart. Keep them fixed in this position.

Imagine a vertical line which crosses the parallel lines … Think to yourself: How many points of intersection can I see? … Can you see eight right angles? Count them to make sure.

Now imagine a point on the vertical line, halfway between the parallel lines. Imagine pushing a pin into the vertical line so that it is fixed at this point … Now **very slowly** rotate the vertical line clockwise about the point fixed by the pin.

As you slowly rotate the line, think to yourself: What happens to the points where it crosses the parallel lines? … What happens to the right angles?

Continue rotating very slowly until the line is sloping at 45°, then stop … How many angles of 45° can you see? ... What is the size of the other angles?

Keeping the sloping line (transversal) fixed at 45°, make the parallel lines move **very slowly** apart, keeping them still horizontal and parallel.

As you slowly move (translate) the lines apart, think to yourself: What happens to the points of intersection? … What happens to the angles at these points – do they change in size?

Now push the parallel lines slowly towards each other … What happens to the points of intersection? … What happens to the angles at these points – do they change in size?

Pupils find it difficult to see shapes within sets of intersecting lines. They are sometimes reluctant to extend the lines forming a shape and this is often needed for a proof. The following two exercises practise extending lines forming shapes while maintaining a fixed picture of the original shape, focusing on vertically opposite angles at points of intersection.

Visualisation 3: Extending the sides of a triangle

Imagine a triangle – make sure all the sides and angles are different.

Think to yourself: Which interior angle is the biggest? … Which is the smallest?

Check the vertices of the triangle. Pin these in place – they **must not move**. Fix your mind on one of the sides and allow the line to grow in both directions. Remember the vertices stay still so the line extends beyond the vertices at each end.

Think to yourself: How would I describe what I can see now?

Fix your mind on another of the sides and allow the line to grow in both directions. Remember the vertices stay still so the line extends beyond the vertices at each end.

Finally do the same to the last side. Allow the line to grow past the two vertices, remembering that these are pinned in position and do not move as the line grows.

Look at the three vertices of the triangle. The growing lines have produced something like an X shape at each vertex. Look at one of the vertices … Can you see four angles formed around this X shape?

Shade your triangle so that you can easily see the original shape.

Concentrate on the smallest interior angle of your triangle … Look at the intersection now formed here … Can you see another angle at the intersection which will be the same size as this smallest angle?

Visualisation 4: Extending the sides of a parallelogram

Imagine a parallelogram … Think to yourself: Roughly how big is each interior angle? … Which is the bigger pair of angles? … Which is the smaller?

Check the vertices of the parallelogram. Pin these in place – they must not move. Fix your mind on one of the sides and allow the line to grow in both directions. Remember the vertices stay still so the line extends beyond the vertices at each end.

Think to yourself: How would I describe what I can see now?

Fix your mind on the side parallel to the line you have extended and allow it to grow in both directions. Remember the vertices stay still so the line extends beyond the vertices at each end.

Do the same for the other pair of sides. Allow the lines to grow past the vertices, remembering that these are pinned in position and do not move as the lines grow.

Look at the vertices of the parallelogram. The growing lines have produced something like an X shape at each vertex. Look at one of the vertices … Can you see four angles formed around this X shape?

Shade your parallelogram so that you can easily see the original shape.

Concentrate on the smaller pair of interior angles … Look at one of the intersections now formed here … Can you see another angle at this intersection which will be the same size as this smallest angle?

The next two examples take the opposite approach to visualisations 3 and 4, starting with sets of lines and visualising the shapes within the intersections.

Visualisation 5: Three intersecting lines

Imagine three very long straight lines. Let them wander freely, changing direction and position, crossing (intersecting) each other but remaining straight.

Now fix the lines so that they all cross (intersect) at the same point. How many angles can you see at this point? Count them to yourself.

Now arrange the lines so that they cross (intersect) in three different points. What shape do they enclose?

Concentrate on the enclosed shape. Roughly how big is each interior angle?

Concentrate on **one** of the points of intersection. How many angles surround this point? Are any of them equal?

Visualisation 6: Two pairs of parallel lines

Imagine a pair of parallel lines, infinitely long. Make them horizontal, not too far apart.

Imagine another pair of parallel lines; arrange them vertically to cross the first pair at right angles … What shape is enclosed by the two pairs of lines? … How many right angles can you see? Count them.

Keeping the horizontal lines fixed as before, pull the vertical lines apart. Can you enclose a long thin rectangle?

Now slowly push the vertical lines together … Stop when they enclose a square … What shape do you get if you push them even closer together?

Still keeping the horizontal pair of lines fixed, can you rotate the other pair so that the lines enclose a parallelogram?

Now slowly push the sloping lines together, or pull them apart, until the lines enclose a rhombus. How many angles can you see? Count them. Are they all the same?

This last example might lead to a discussion about the congruence of shapes.

Visualisation 7: Halving a rectangle

Imagine a rectangle.

Now imagine one of the lines of symmetry of that rectangle. Ask yourself: How many times does the line intersect the rectangle? … How many right angles can I see?

Imagine pushing a pin right into the centre of the rectangle, so that the line of symmetry is fixed at the centre of the rectangle.

Now, keeping the rectangle in the same place, **very slowly** rotate the line anticlockwise about the point fixed by the pin. Notice this is no longer a line of symmetry …

Stop rotating your line. Imagine cutting the rectangle along the line and pulling the two bits of the rectangle apart. What can you say about the two new shapes?

As a follow-up, you may want to consider this alternative to class discussion.

- *Without drawing any diagrams, write down all the things you can now say about the two new shapes.*
- *Compare notes with a friend. Convince them of your ideas, using diagrams if you wish.*
- *If you had started with a rhombus, what would have changed? What would be the same?*

Prompts for main activities in phase 1

Build-ups

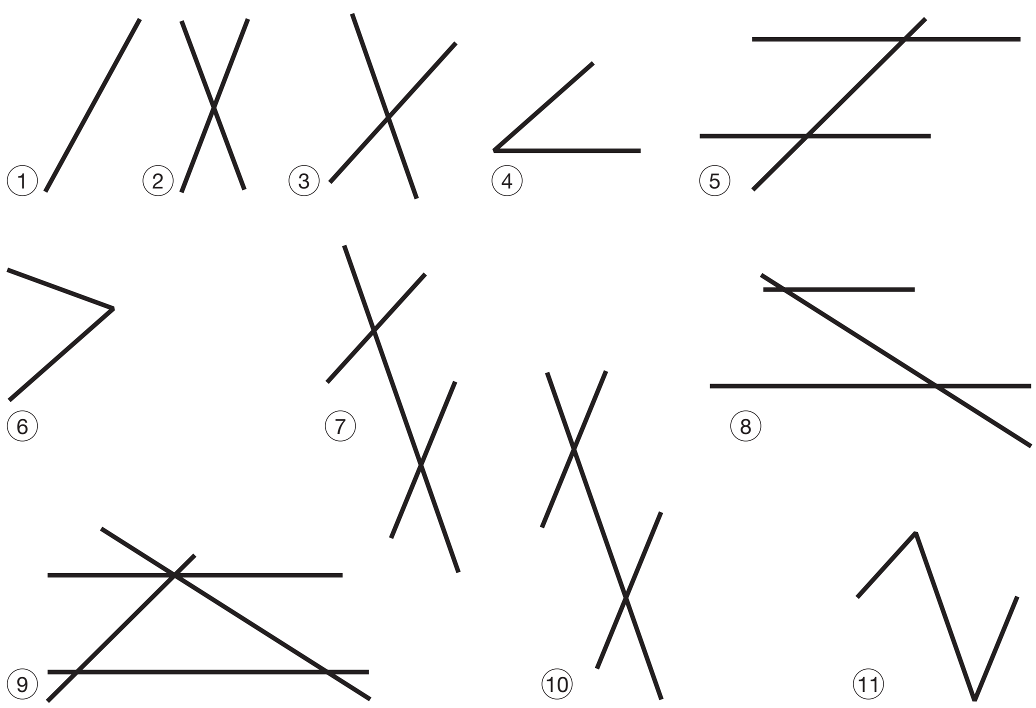
These activities aim to support pupils in developing a structure to their reasoning as they work from pictures to words.

Resources: Overlays for ‘build-ups’ copied on to acetates (for the teacher) and on to tracing paper (for pupils). Note that some of these will need to be cut in half so that each figure is separate.

The essence of this activity is that acetates representing given facts are used to ‘build up’ new geometrical properties.

- Each layer of the build-up contains a **given fact** to be shown on the diagram; for example, angles on a straight line add to 180°.
- As these facts are collated, certain **conventions** are introduced to secure the notation on the diagram; for example, arrows are marked on parallel lines.
- General **definitions** are also established. For example, alternate angles are defined by their positions relative to a pair of lines and a transversal, rather than becoming known only by their equality when the pair of lines are parallel.
- As the layers are built up, the ‘givens’ are noted; for example, ‘the first layer we are given shows that pairs of alternate angles on parallel lines are equal’.
- This is gradually abbreviated as the facts themselves and the conventions of notation become familiar; for example:
  - Given: pairs of alternate angles on parallel lines are equal.
  - Given: equal alt ∠s on // lines.
- The final fact which is established through the build-up is ‘inked in’ and becomes a layer which can be used in subsequent build-ups.

Build-up figures 1 to 11



Creating images

The activity involves whole-class demonstration of the build-ups using acetates. Pupils then repeat the process, initially reconstructing the previous proof and, later, completing other build-ups using tracing paper in small groups.

Successive figures (drawn from the selection shown here) are carefully positioned on top of earlier figures to create the ‘picture’. The features illustrated should be ‘inked in’ clearly on the overlays, showing necessary conventions.

The build-up of known facts needs to be clearly articulated by pupils. They will need practice!

When the build-up is complete, it is a good idea to use a final sheet to replace the overlays. Consider this as a ‘freeze frame’ of the picture you have just built up.

At this final stage conventions will need to be added and definitions of terms reinforced. Involve pupils in this process as this will help them to reconstruct the oral version of the proof.

Seven build-ups are described below.

Build-up number	**Figures to use**	**Givens**	**To establish/prove**
1	fig 1 (2 copies) fig 2	Angles on a straight line add to 180°	To prove that vertically opposite angles are equal
2	fig 2 fig 3 fig 7		To establish the definition of corresponding angles
3	fig 2 (2 copies) fig 3 fig 7 fig 10	The definition of corresponding angles	To establish that pairs of corresponding angles on parallel lines are equal
4	fig 4 fig 6 fig 7 fig 11		To establish the definition of alternate angles
5	fig 10 fig 1	Pairs of corresponding angles on parallel lines are equal Angles on a straight line add to 180°	To prove that pairs of alternate angles on parallel lines are equal
6	fig 5 fig 8 fig 1 fig 9	Pairs of alternate angles on parallel lines are equal Angles on a straight line add to 180°	To prove that the angles in a triangle add to 180°
7	fig 5 fig 8 fig 1 fig 9	Pairs of alternate angles on parallel lines are equal Pairs of corresponding angles on parallel lines are equal Angles on a straight line add to 180°	To prove that the angles in a triangle add to 180°

Developing a written argument

As the image is created, the verbal argument should be recorded. Some suggestions are given in the third column of each build-up. Modelling the process here will assist pupils in their recording of solutions to problems in phase 2 of this unit (and beyond).

The approach used here could be:

1 joint recording of a whole-class demonstration, perhaps a pupil scribing at the class whiteboard;

2 small groups working together recording on a shared small whiteboard or equivalent;

3 reprocessing the recorded argument more concisely and neatly on an individual basis.

Sequencing the build-ups

A choice of sequences is possible and should be made as appropriate to the teaching group.

- The full sequence (build-ups 1–7) provides a complete logical sequence, establishing part of a deductive hierarchy of geometrical facts (see 'Prompts for final plenary in phase 2' **page 30**). Within this sequence corresponding and alternate angles are defined by their relative positions on any pair of lines with a transversal (build-ups 2 and 4); the special case of these angles on pairs of parallel lines is then considered (build-ups 3 and 5).
- A shortened sequence (build-ups 1, 5, 6, 7) works with pupils' existing understanding of the position of corresponding and alternate angles on pairs of parallel lines. Equality of corresponding angles on parallel lines is then accepted as a given and used to prove that alternate angles on parallel lines are equal using build-up 5.

As a longer-term aim it may be helpful to begin from the broader definitions of alternate and corresponding angles in earlier years. Pupils are then aware of the specific properties established by the pairs of lines becoming parallel.

Build-ups in detail

On the next seven pages, each build-up is described in detail.

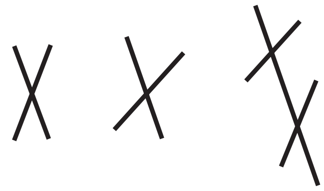
Build-up 1: To prove that vertically opposite angles are equal

This build-up uses figures 1 (two copies) and 2.

Show	Say	Note on paper or board as the layers are built up
Fig 1 on screen. Mark a point on the line and mime drawing an angle arc.	We know that angles on a straight line add to 180°. I am giving you that fact.	Given: angles on a straight line add to 180°
New fig 1 on top of existing fig 1 so that the lines coincide.	Here is another straight line. I can put it directly on top of the original.	
Rotate top fig 1 about the point marked. Rotate a few times to show different angles.	I can rotate it like this. We now have a point of intersection. Can you see four angles formed here? Can you see them changing as I rotate the lines?	
Point to various pairs of adjacent angles.	The angles are in pairs. Some are next to each other. We call these **adjacent angles**.	
Point to a pair and mime the drawing of a straight line or place a straight edge along a line to show them clearly.	What do we know about these pairs of adjacent angles?	Same given: pairs of adjacent angles on a straight line add to 180°
Point to the two pairs of vertically opposite angles.	Some pairs of angles are opposite each other. We call these **vertically opposite angles** (not the most logical term since some don't look vertical at all!).	
Replace with fig 2 as a freeze frame. Mark one pair of vertically opposite angles with letters p and q another pair with letters a and b.	What can we say about the comparative size of these angles? … Can we be rigorous in our argument?	Want to prove that vertically opposite angles are equal Want to prove that $p = q$ and $a = b$
Note: Some pupils may argue this through using a dynamic approach. 'As you rotate it, this angle gets bigger and the vertically opposite one gets bigger by the same amount'. This is good and will help some pupils gain an understanding of the angle relationships. With some persuasion you may also get the following argument.		
Position fig 1 along each straight line as the argument is made. Point out the angle each straight line has in common (in this case angle p).	We know a and p make 180°. We know p and b make 180°. That must mean that a and b are the same size.	$a + p = 180$ $p + b = 180$ $\therefore a = b$ (adjacent ∠s on straight line add to 180°)
Position fig 1 along each straight line as the argument is made. Point out the angle each straight line has in common (in this case angle b).	Repeat the logic for the other pair (p and q).	$p + b = 180$ $b + q = 180$ $\therefore p = q$ (adjacent ∠s on straight line add to 180°)
Perhaps show two copies of fig 1 again.	Will this argument work for all intersecting lines? Remember when we rotated them, did it matter where we stopped?	Proved: vertically opposite angles are equal

Build-up 2: To establish the definition of corresponding angles

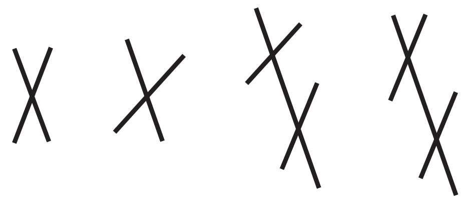
This build-up uses figures 2, 3 and 7.



Show	Say	Note on paper or board as the layers are built up
Fig 2 on screen.	Here are two intersecting lines.	
Fig 3 on screen.	Here is a different pair of intersecting lines.	
Slide fig 2 and fig 3 so that one line is produced (as shown in fig 7).	We can slide these so that one of the straight lines is continuous through both points of intersection.	
Mime the drawing of just one transversal and show the two lines.	Can you see the 'join'? This could have started as one straight line crossing the other pair of lines. (We would call this a transversal.)	
Point to or 'dot' the appropriate angles.	Can you see the 'top' angle at each of the points of intersection? We say that these are in the same position as each other relative to the intersecting lines.	
Pupils point to or mark pairs of angles.	Which other angles are in the same position as each other relative to the intersecting lines?	
Fig 7 on screen. Mark angles at each point of intersection with letters (*a*, *b*, *c*, *d* and *p*, *q*, *r*, *s*).	We have been finding pairs of **corresponding angles**. Use the letters to state pairs of corresponding angles.	By pairs of **corresponding angles** we mean angles in the same position as each other relative to the intersecting lines.
Fig 2 and fig 3 replaced on screen with just one pair of corresponding angles marked.	Do you think that pairs of corresponding angles are the same size?	
Translate fig 3 along the 'transversal' to sit on top of fig 2. Point to the pair of corresponding angles.	We can see that these corresponding angles are not the same size.	
Fig 7 on screen with letters still marking all eight angles.	Who can remind me of another pair of corresponding angles and convince me that they are not equal?	

Build-up 3: To establish that pairs of corresponding angles on parallel lines are equal

This build-up uses figures 2 (two copies), 3, 7 and 10.

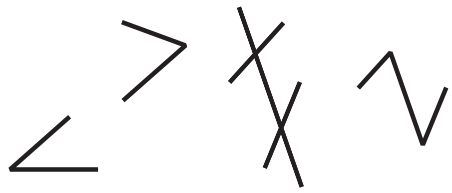


Note: This process does not *prove* that corresponding angles on parallel lines are equal. Part way through we effectively have to establish the meaning of parallel lines. It is important nonetheless to convince pupils of their equality, used as a given in our system.

Show	Say	Note on paper or board as the layers are built up	
Fig 7 on screen.	Do you remember our discussion about corresponding angles?		
Fig 2 and fig 3 on screen.	Do you remember how we generated this final diagram? Under what circumstances would we have a diagram with all pairs of corresponding angles equal?		
Pupils may suggest: Replace fig 3 by a duplicate fig 2 on top of first fig 2 to show identical.	Why do the figures have to be identical?		
Slide top fig 2 so that one line is produced (as shown in fig 10).	We can slide these apart so that one of the straight lines is continuous through both points of intersection.		
Show the two lines. Mime the drawing of just one transversal.	Can you see the 'join'? This could have started as a pair of lines crossed by one transversal.		
Mime drawing a pair of lines at a variety of different angles. Slide fig 2 back on top of itself to show what is special about this case.	Could we draw the pair of lines anyhow? … What is special about the way we draw this pair of lines?		
Slide top fig 2 back out so that one line is produced (as shown in fig 10). Mark arrows to indicate parallel lines. Identify the transversal.	The lines as built up here produce a pair of **parallel lines**.	Given: parallel lines are established here	
Slide figures back together and apart again to confirm this. Show with the same letter a pair of identical angles.	We say that this is a pair of corresponding angles because they are in the same relative position. Now we also know that they are equal because they are on a pair of parallel lines.	Establish: pairs of corresponding angles on parallel lines are equal	
Replace with fig 10 as a 'freeze frame'. Ask a few pupils to mark up the diagram and talk through their chain of reasoning for the equality of some pairs of angles.	There are 8 angles at these two points of intersection. How many different sizes of angle are there? The simplest way of recording this is to use the same symbol for angles known to be equal.	Convention: when angles are equal we mark them with the same symbol	
Draw attention to one point of intersection first. Mark equal angles (letters *g* and *h*).	Here we have two pairs of vertically opposite angles. We need to say why we are marking angles with the same letter. We usually put the reason in brackets.	One pair marked as *g*, the other pair marked as *h*	(vertically opposite ∠s equal)
Now look at the second point of intersection. Mark equal angles (letters *g* and *h*).	We can use the same pair of symbols to mark up this second point of intersection. Here we have pairs of angles which correspond to those marked *g* and *h*.	One pair marked as *g* the other as *h*	(corr ∠s on // lines equal)

Build-up 4: To establish the definition of alternate angles

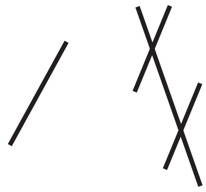
This build-up uses figures 4, 6, 7 and 11.



Show	Say	Note on paper or board as the layers are built up
Fig 4 on screen.	Here is an angle formed by two lines.	
Fig 6 on screen.	Here is another angle formed by two lines.	
Slide fig 4 and fig 6 so that one common line is formed and a zig-zag is produced (as shown in fig 11).	We can slide these together to form a ‘zig-zag’.	
Mime the drawing of the three lines forming the zig-zag.	Can you see the ‘join’? This could have started as one diagram.	
Point to or ‘dot’ the appropriate angles.	Can you see the acute angles contained in the zig-zag?	
Mime the drawing of the transversal and point to the alternate sides.	We can describe these angles as being formed by the top line and the transversal and by the bottom line and the transversal. But they are on alternate sides of the transversal. We call these **alternate angles**.	By pairs of **alternate angles** we mean angles formed on either side of a transversal crossing two straight lines and contained between the two straight lines
Fig 7 on screen.	Can you pick a zig-zag out of this diagram? Mark the alternate angles.	
Fig 4 and fig 6 replaced on screen with just the pair of alternate angles marked.	Do you think that pairs of alternate angles are the same size?	
Rotate fig 6 to sit on top of fig 4. Point to the pair of marked angles.	We can see that these angles are not the same size.	
Fig 7 on screen. Overlay fig 4 and fig 6.	Who can remind me of why this is labelled as a pair of alternate angles and convince me that they are not equal?	

Build-up 5: To prove that pairs of alternate angles on parallel lines are equal

This build-up uses figures 1 and 10.

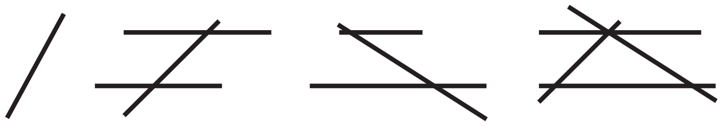


Note: An alternative proof could be developed using ‘vertically opposite angles’ in the sequence in place of ‘angles on a straight line’.

Show	Say	Note on paper or board as the layers are built up	
Fig 10 on screen. Pupils mark arrows to show the parallel lines and one pair of corresponding angles using the same letter.	We know that pairs of corresponding angles on parallel lines are equal. I am giving you that fact.	Given: pairs of corresponding angles on parallel lines are equal	
Pupils point to and describe in full the position of the pair of corresponding angles. Pupils indicate (but do not mark) other pairs of corresponding angles.	Do you remember our discussion about corresponding angles? We gave them their name rather than have to make a lengthy description of their position.	One pair marked as g	(corr ∠s on // lines equal)
Mark with x and y two alternate angles (choose angles which are different from the corresponding angles marked).	Now look at this pair of angles. How could we describe their relative positions?	Angle x is alternate to angle y	
Pupils indicate (but do not mark) another pair of alternate angles.	They are on different/alternate sides of the transversal. Do you remember this is a pair of alternate angles?		
	Can we say anything about the comparative size of the alternate angles we have marked? … We must be precise in our reasoning …	Want to prove that pairs of alternate angles on parallel lines are equal Want to prove that $x = y$	
Position fig 1 on one of the straight lines formed by two of the angles marked.	What fact am I giving you?	Given: ∠s on straight line add to 180°	
Position fig 1 in turn on each of the straight lines formed by two of the angles marked.	What does this show you about each pair of angles?	$g + x = 180$ $g + y = 180$	(adjacent ∠s on straight line add to 180º)
	What can we deduce from these two statements? … Angle g is common to both equations so the values of x and y must be the same.	$\therefore x = y$	
Remove any overlays. Show a clean version of fig 10. Ask a few pupils to label the diagram and talk through their chain of reasoning for each pair of alternate angles.			
Pupils could sketch different sets of parallel lines and transversals.	Remember that we could slide parallel lines apart or together and change the angle of the transversal. Would our argument still be true?	Proved: pairs of alternate angles on // lines are equal	

Build-up 6: To prove that the angles in a triangle add to 180°

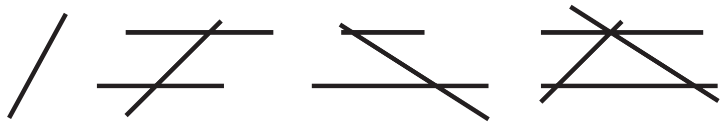
This build-up uses figures 1, 5, 8 and 9.



Show	Say	Note on paper or board as the layers are built up	
Fig 5 on screen. Mark parallel lines with arrows.	We know that pairs of alternate angles on parallel lines are equal. I am giving you that fact.	Given: pairs of alternate ∠s on // lines are equal	
Mime pulling the parallel lines apart and label the equal pair of alternate angles (choose the acute pair).	Do you remember the discussion about alternate angles? We could pull the parallel lines further apart or squash them closer and still have an equal pair of alternate angles.	One pair marked as *k*	(alt ∠s on // lines equal)
Fig 8 on top of fig 5 but not forming triangle. Mark parallel lines with arrows.	This is the same given fact but the transversal has a different slope and is in the opposite direction.		
Label new pair of equal alternate angles on fig 8 using a different letter to first pair on fig 5 (choose the acute pair).	Let's recap the facts so far … We have two pairs of parallel lines which are drawn at exactly the same distance apart. We have one pair of alternate angles already marked which are equal to each other. We now have a second (different) pair to mark which are equal to each other.	One pair marked as *m*	(alt ∠s on // lines equal)
Slide fig 8 so that a triangle is formed. (Note there are two possible ways of doing this.)	Can you see how a triangle is formed?		
Point to the three interior angles. Mark remaining angle as *p*.	We want to know what we can say about the sum of the angles inside the triangle using the facts we have just built up.	Want to prove that ∠s in a Δ add to 180° Want to prove that $k + m + p = 180$	
Point to the three marked angles on the straight line. Overlay fig 1 to be clear about which straight line is being used.	There is one more fact here that we have been given on a previous occasion. What is it?	Given: angles on a straight line add to 180°	
Point to the three angles on the line (*k*, *p* and *m*). Point to the pairs of alternate angles to convince that the same three angles appear as the sum of the interior angles.	Could someone use the two facts we have been given to prove that the angles in a triangle must add up to 180°?	$k + m + p = 180$ same ∠ *k* marked in Δ same ∠ *m* marked in Δ Proved: the angles in a Δ add to 180°	(∠s on a straight line add to 180°) (alt ∠s on // lines equal)
Replace with fig 9 as a 'freeze frame'. A few pupils mark up and point to diagram and talk through their chain of reasoning.			

Build-up 7: To prove that the angles in a triangle add to 180°

This build-up uses figures 1, 5, 8 and 9.



Show	Say	Note on paper or board as the layers are built up	
Fig 5 on screen. Mark parallel lines with arrows.	We know that pairs of alternate angles on parallel lines are equal. I am giving you that fact.	Given: pairs of alternate ∠s on // lines are equal	
Mime pulling the parallel lines apart and label equal alternate angles (choose the acute pair).	Do you remember the discussion about alternate angles? We could pull the parallel lines further apart or squash them closer and still have an equal pair of alternate angles.	One pair marked as x	(alt ∠s on // lines equal)
Fig 8 on top of fig 5 but not forming triangle. Mark parallel lines with arrows.	We know that pairs of corresponding angles on parallel lines are equal. I am giving you that fact.	Given: pairs of corresponding ∠s on // lines are equal	
On fig 8 label equal pairs of corresponding angles (choose the acute pair on left of transversal).	Let's recap the facts so far … We have two pairs of parallel lines which are drawn at exactly the same distance apart. We have one pair of alternate angles already marked which are equal to each other. We now have a pair of corresponding angles to mark which are equal to each other.	One pair marked as y	(corr ∠s on // lines equal)
Slide fig 8 so that a triangle is formed	Can you see how a triangle is formed?		
Point to the three interior angles. Mark remaining angle as z.	We want to know what we can say about the sum of the angles inside the triangle using the facts we have just built up.	Want to prove that ∠s in a Δ add to 180° Want to prove that $x + y + z = 180$	
Point to the three marked angles on the straight line. Overlay fig 1 to be clear about which straight line is being used.	There is one more fact here that we have been given on a previous occasion. What is it?	Given: ∠s on a straight line add to 180°	
Point to the three angles on the line (x, y and z).	Could someone use the two facts we have been given to prove that the angles in a triangle must add up to 180°?	$x + y + z = 180$	(∠s on a straight line add to 180°)
Point to the pair of alternate angles and the pair of corresponding angles to convince that the same three angles appear as the sum of the interior angles.		same ∠ x marked in Δ same ∠ y marked in Δ	(alt ∠s on // lines equal) (corr ∠s on // lines equal)
Replace with fig 9 as a 'freeze frame'. A few pupils point to diagram and talk through their chain of reasoning.		Proved: the ∠s in a Δ add to 180°	

Prompts for main activities in phase 2

Solving problems

One approach for teaching geometrical problem-solving skills is:

1 **Select suitable problems from the problem bank.**
The problem bank (mini-pack, pages 32–36) is organised in three sections:
- problems where a labelled diagram is given;
- problems where a diagram has to be drawn – for each of these an alternative wording is suggested, providing direct instructions for drawing the diagram;
- a couple of more extended problems to investigate.

2 **Model the task with the whole class.**
Notes for modelling the task are detailed below.

3 **Set up independent work in pairs.**
The following is a suggested way of organising the work of pairs of pupils.

In the following problems you are expected to explain each step of your reasoning. You are allowed to take the following facts as given:

- Angles on a straight line add to 180°.
- Vertically opposite angles are equal.
- Pairs of corresponding angles on parallel lines are equal.
- Pairs of alternate angles on parallel lines are equal.
- The interior angles of a triangle add to 180°.
- The interior angles of an n-sided polygon add to $(n – 2)180°$.
- Angles opposite the equal sides of an isosceles triangle are equal.
- In a triangle with two equal angles, the sides opposite the equal angles are equal and the triangle is isosceles.

Nothing else must be assumed!

Work in pairs:

- **Solve the problem** on your own if you can. You may need to draw a diagram and label some points and angles.
- **Compare notes:** take it in turns to explain your thinking to each other – be ready to act as a 'doubter' and question your partner so that they make everything clear.
- **Agree on a solution**, talk through the steps and carefully write out the proof, making sure that a reason is given for each step. Read and, if necessary, re-draft your explanation.
- **Present the alternative,** if the two of you have taken different steps to get to the solution.

Modelling the task and supporting pupils' explanations

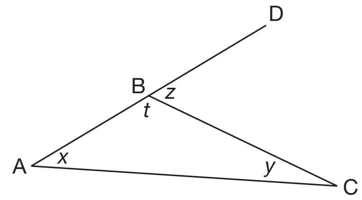
Stages in the process	Support strategies
Clarify the task. Draw a diagram. State what is to be proved. Restate in terms of simple labels on diagram.	 Diagram provided. Teacher clarifies. *Pupils restate in terms of labels.*
Build up the chain of reasoning. Make statements of facts. Give reasons for the facts.	 Teacher provides facts. *Pupils add reasons for the facts* ***or****, in shorter chains, order a set of jumbled statements.*
Deduce some information. Calculate or solve where needed.	 Teacher constructs the calculation. *Pupils conclude calculation.*
Conclude the argument. Refer back to original task statement.	

It is suggested that in the first lesson of phase 2 you model the task with the whole class, emphasising the four stages in the process. The following is an example of this process, using problem 6 from the problem bank.

> **6** Prove that the exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices.
>
> Alternative wording:
>
> Draw ΔABC. Extend side AB beyond vertex B to point D.
>
> Prove that $\angle$**CBD =** $\angle$**ACB +** $\angle$**CAB**

- The 'formal' question is read out, brief thinking time allowed and visualisation encouraged.
- Pupils use whiteboards to sketch the diagram as the alternative wording is read out.
- Diagrams are compared and 'tidied up' (they will not all be identical).
- A pupil draws a diagram on the board or OHP. Labels are agreed and added.
- Pupils discuss and explain their reasoning orally or using jottings.
- Steps in the argument are presented in writing:
  - each step on a separate line;
  - reason in brackets after each statement (including 'given');
  - shorthand such as ∴, Δ, //.

Want to prove: $\angle CBD = \angle ACB + \angle CAB$
$z = x + y$



Label $\angle ABC = t$

$x + y + t = 180$ (interior angles of ΔABC)
$z + t = 180$ (angles on straight line ABD)
$\therefore z = x + y$
$\angle CBD = \angle ACB + \angle CAB$

This is a standard result, which could be added to the list of givens.

Varying the expectations of pupils' written explanations

The aim is to improve pupils' skills of giving logically ordered explanations, with reasons. A clear written explanation is much more difficult to give than an oral one. Pitch expectations appropriately for the pupils – sufficiently high to challenge them, but not so high as to be demotivating.

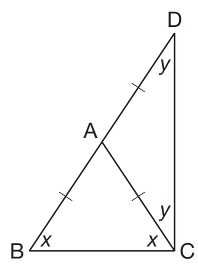
For clarity and ease of explanation it is a good idea to adopt the following conventions when marking angles on diagrams:

- Use a single lower-case letter to represent the size of an angle.
- If two angles are known to be equal use the same letter, giving reasons for doing so in the written argument.

To illustrate how expectations can be varied, consider problem 14 from the problem bank:

> **14** A is a vertex of an isosceles triangle ABC in which AB = AC. BA is extended to D, so that AD is equal to BA. If DC is drawn, prove that $\angle BCD$ is a right angle.
>
> Alternative wording:
>
> Draw an isosceles triangle ABC with AB = AC. Extend side BA beyond vertex A to a point D so that BA = AD. Join D to C. Prove that $\angle$**BCD = 90°**

Want to prove: $\angle BCD = 90°$
$x + y = 90$



A developing written argument

Many pupils will give a simplified explanation and this may represent considerable progress for them. Encourage them to include the key points and give clear reasons.

The angles marked x are equal (isosceles ΔABC).
The angles marked y are equal (isosceles ΔACD).

Add all three angles in ΔBCD:

$x + (x + y) + y = 180$
$2x + 2y = 180$
$x + y = 90$

So $\angle BCD = 90°$

A complete written argument

This explanation is about as rigorous as it could get with the most capable pupils.

$\angle ABC = \angle ACB$	(angles opposite equal sides of an isosceles Δ)
Label these angles x	
$AC = AB$	(given)
$AD = AB$	(given)
$\therefore AD = AC$	(both equal AB)
$\therefore \Delta ADC$ is isosceles	(two equal sides)
$\therefore \angle ACD = \angle ADC$	(angles opposite equal sides of an isosceles Δ)
Label these angles y	
In ΔBCD:	
$x + (x + y) + y = 180$	(angles in a Δ)
$2x + 2y = 180$	
$x + y = 90$	
$\angle BCD = x + y$	
$\therefore \angle BCD = 90°$	

Prompts for first plenary in phase 1

Alice's Adventures in Wonderland

Resources: CD track 8
Copy of text on OHT or handout for pupils (see resource sheet)

Play CD track 8 or read the passage from the trial of the Knave of Hearts in *Alice's Adventures in Wonderland* (Ch XII, Alice's evidence). Display a copy of the text as an OHT or provide pupils with handouts.

> *'There's more evidence to come yet, please your Majesty,' said the White Rabbit, jumping up in a great hurry: 'this paper has just been picked up.'*
>
> *'What's in it?' said the Queen.*
>
> *'I haven't opened it yet,' said the White Rabbit, 'but it seems to be a letter, written by the prisoner to – to somebody.'*
>
> *'It must have been that,' said the King, 'unless it was written to nobody, which isn't usual, you know.'*
>
> *'Who is it directed to?' said one of the jurymen.*
>
> *'It isn't directed at all,' said the White Rabbit: 'in fact, there's nothing written on the* outside*.' He unfolded the paper as he spoke, and added, 'It isn't a letter, after all: it's a set of verses.'*
>
> *'Are they in the prisoner's handwriting?' asked another of the jurymen.*
>
> *'No, they're not,' said the White Rabbit, 'and that's the queerest thing about it.' (The jury all looked puzzled.)*
>
> *'He must have imitated somebody else's hand,' said the King. (The jury all brightened up again.)*
>
> *'Please your Majesty,' said the Knave, 'I didn't write it, and they can't prove that I did: there's no name signed at the end.'*
>
> *'If you didn't sign it,' said the King, 'that only makes the matter worse. You* must *have meant some mischief, or else you'd have signed your name like an honest man.'*
>
> *There was a general clapping of hands at this: it was the first really clever thing the King had said that day.*

After the reading:

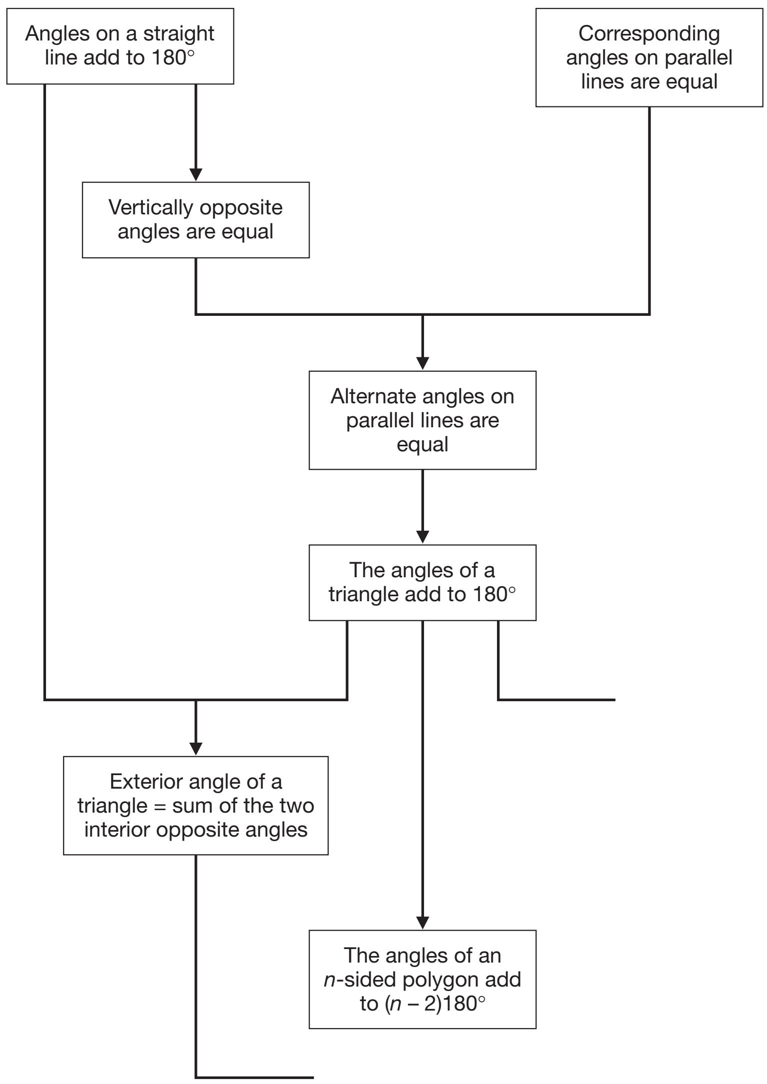
- Ask the class: 'What do you find illogical about this passage?'
- Referring to the written text (on OHT or handouts), get the class to identify particular statements that are unjustified – either assumptions or conclusions.
- Ask: 'Why did I read/play this passage to you?'
- Draw out links with the main part of the lesson using logical argument based on reasonable assumptions ('givens', e.g. angles on a straight line sum to 180°) to reach sound conclusions (e.g. vertically opposite angles are equal).
- Explain that this unit of work is about developing reasoned arguments in geometry – being clear about what is given and setting out the steps of the argument with reasons.

Prompts for final plenary in phase 2

Flowchart of logical reasoning

Resources: Prepare an OHT or wall chart of the resource sheet 'Start of a flowchart of logical reasoning'.

Start of a flowchart of logical reasoning

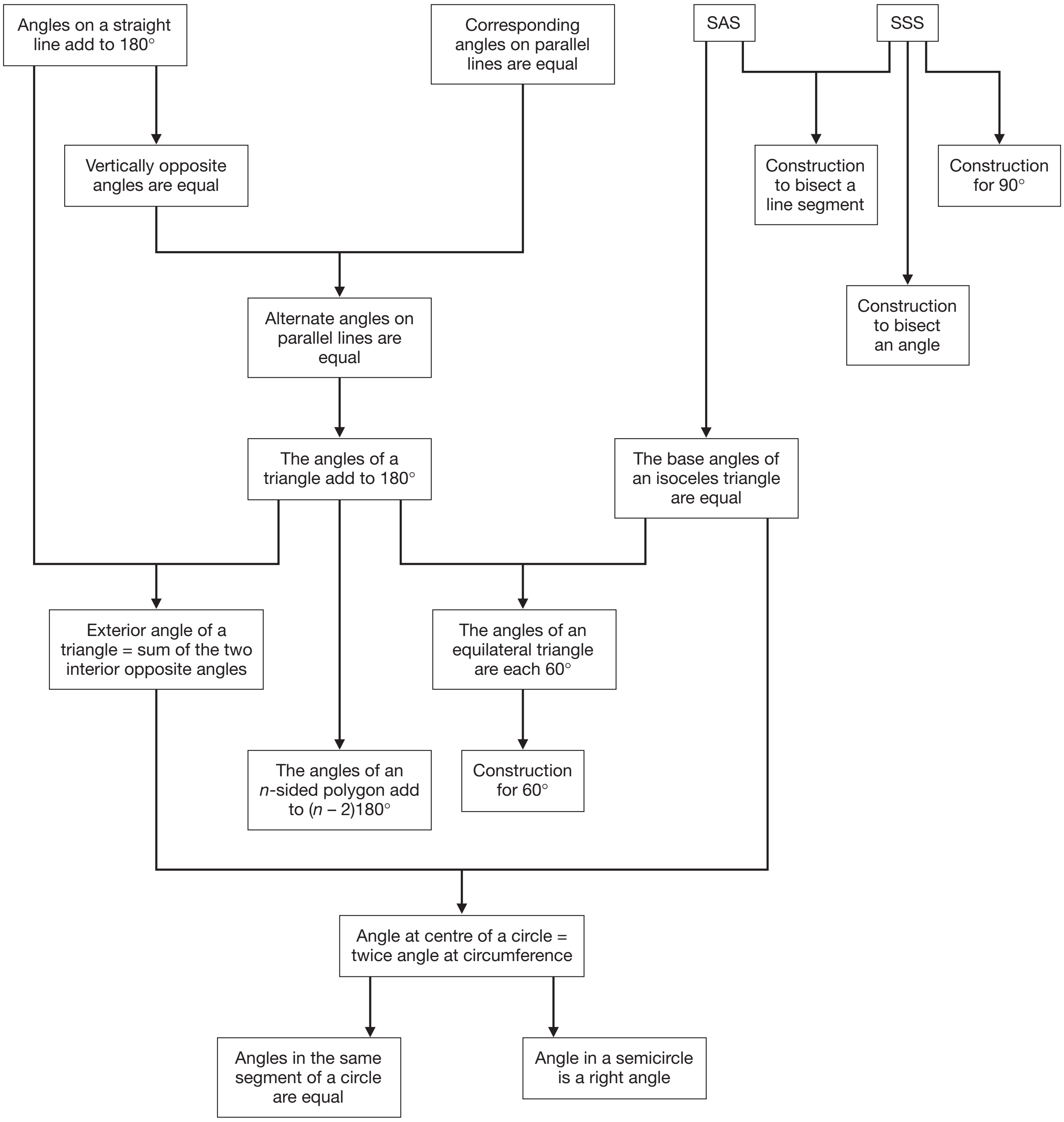


Ask pupils to explain the chart by posing questions such as:

- Why are the geometrical facts organised in this particular order, starting at the top and working down the chart?
- What are the links and arrows intended to show?
- Can you explain how to use the given facts to deduce that (a) vertically opposite angles are equal (b) alternate angles are equal?
- Are the facts about triangles and polygons correctly placed in the chart?

Point out that there are some unattached links. As more results are developed, perhaps in Key Stage 4, the chart will be extended. (As a matter of general interest, rather than for showing to pupils at this stage, a more extended chart is shown opposite.)

Extended flowchart of logical reasoning

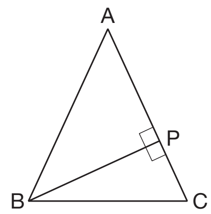


Adapted from 'Starting points and end points', an article by Tony Barnard in *Mathematics in School*, 31(3), May 2002 (Mathematical Association)

Problem bank

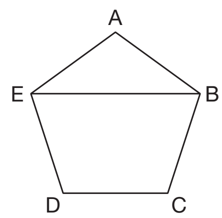
Problems with diagrams

1 In the isosceles triangle shown, AB = AC. From B, a line BP has been drawn to meet the opposite side AC at right angles. Prove that $\angle PBC = \frac{1}{2}\angle CAB$.

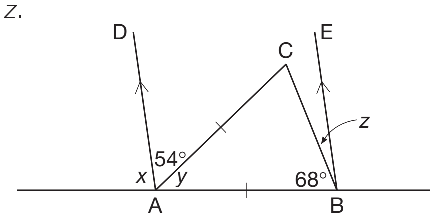


2 The diagram shows a regular pentagon ABCDE. Calculate $\angle ABE$, explaining each step of your reasoning.

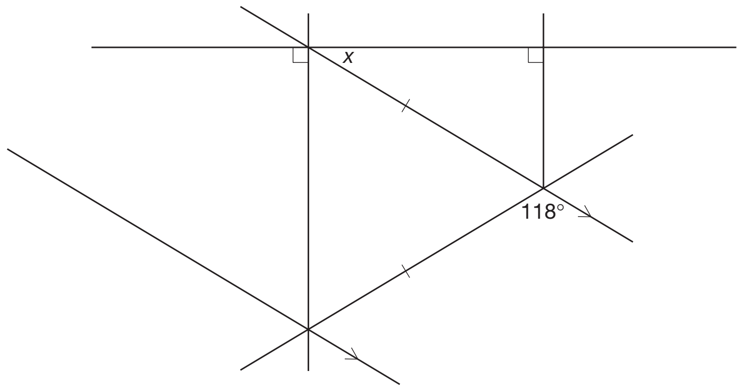
Suppose you started with a regular hexagon? Or a regular *n*-gon?



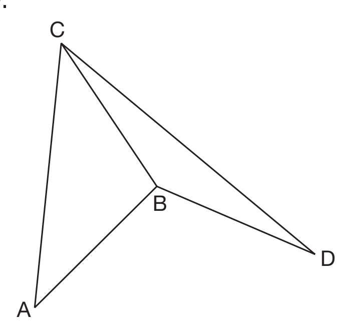
3 In the diagram, ABC is an isosceles triangle with AB = AC, and AD is parallel to BE. Find the values of x, y and z.



4 Find the value of x in the diagram below. (Label the vertices and angles, to help you explain the steps in your reasoning.)



5 ABC is an isosceles triangle with AB = BC. CBD is an isosceles triangle with BC = BD. Prove that $\angle ABD = 2\angle ACD$.



Problems where a diagram must be drawn (with alternative wordings)

6 Prove that the exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices.

Alternative wording:
Draw ∆ABC. Extend side AB beyond vertex B to point D. Prove that **∠CBD = ∠ACB + ∠CAB**.

7 Prove that any straight line drawn parallel to the non-equal side of an isosceles triangle makes equal angles with the other sides.

Alternative wording:
Draw an isosceles triangle ABC with AB = AC. Draw any straight line through the triangle parallel to the side BC. Prove that **the straight line makes equal angles with AC and AB**.

8 From any point on the bisector of an angle a straight line is drawn parallel to one arm of the angle. Prove that the triangle thus formed is isosceles.

Alternative wording:
Draw angle ABC and a line BD bisecting that angle. Draw a straight line DE so that DE is parallel to AB and meets BC at F. Prove that **∆BDF is isosceles**.

9 From X, a point on the side BC of an isosceles triangle ABC (where AB = AC), a straight line is drawn at right angles to BC, cutting AB at Y and CA extended at Z. Prove that the triangle AYZ is isosceles.

Alternative wording:
Draw an isosceles triangle ABC with AB = AC. Extend side CA beyond vertex A. Choose a point X on BC closer to B than C. Draw a straight line at X at right angles to BC, intersecting AB at Y and the extended side CA at Z. Prove that **∆AYZ is isosceles**.

10 If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, prove that the triangle is isosceles.

Alternative wording:
Draw a triangle ABC, extend the side AB beyond B and mark the exterior angle at B. Draw a straight line BD which bisects the exterior angle at B. You are given that the line BD is parallel to AC. Prove that **∆ABC is isosceles**.

11 AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected. If through any point Y on DC a straight line XYZ is drawn parallel to AB and meeting the bisectors at X and Z, prove that XY is equal to YZ.

Alternative wording:
Draw a line AB and mark a point D part way along AB. Draw any line DC. Draw a line DE so that it bisects ∠ADC. Draw a line DF so that it bisects ∠BDC. Draw a line parallel to AB and intersecting DE at X, DC at Y, DF at Z. Prove that **XY = YZ**.

12 If a straight line meets two parallel straight lines and the two interior angles on the same side are bisected, prove that the bisectors meet at right angles.

Alternative wording:
Labelling lines from left to right draw a line AB and draw a second line CD parallel to AB. Draw a transversal intersecting AB at E and CD at F. Draw a line which bisects ∠AEF and another line which bisects ∠CFE. Prove that **these two lines intersect at right angles**.

13 Prove that the angle formed by the intersection of the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining two angles.

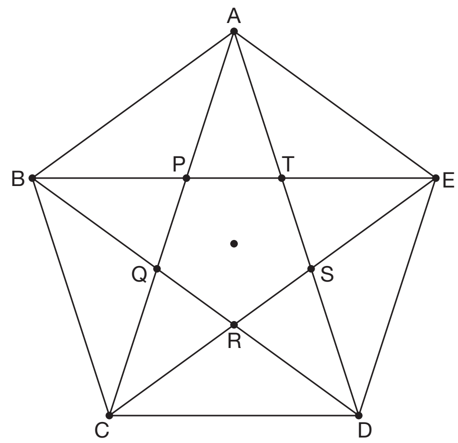
Alternative wording:
Draw a quadrilateral ABCD (label the vertices in order). Draw the line AE so that ∠DAB is bisected. Draw the line DF so that ∠CDA is bisected. Label the point G where AE and DF intersect. Prove that **∠DGA =** $\frac{1}{2}$(**∠ABC + ∠BCD**).

14 A is a vertex of an isosceles triangle ABC, in which AB = AC. BA is extended to D, so that AD is equal to BA. If DC is drawn, prove that ∠BCD is a right angle.

Alternative wording:
Draw an isosceles triangle ABC with AB = BC. Extend side BA beyond vertex A to a point D so that BA = AD. Join D to C. Prove that **∠BCD = 90°**.

Extended problems to investigate

Pentagrams

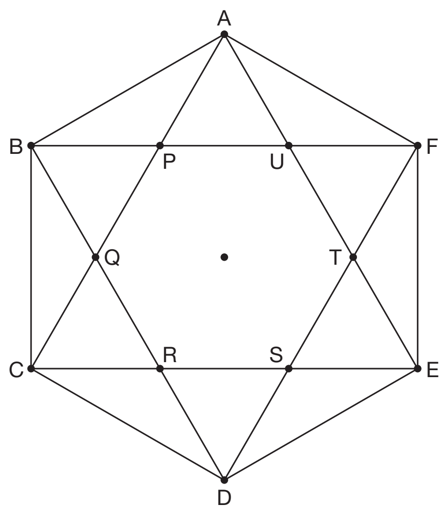


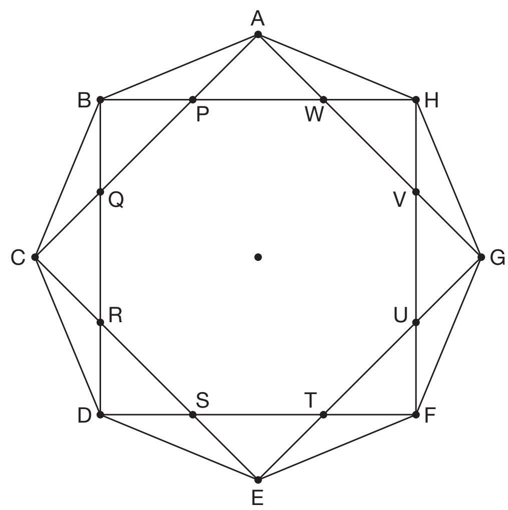
ABCDE is a regular pentagon. A regular pentagram has been formed, and the intersections marked as P, Q, R, S, T.

- How many pairs of parallel lines can you find?
- How many pairs of perpendicular lines?
- Can you find all the angles in the diagram?
- How many different types of triangle can you find?
- How many different types of quadrilateral?

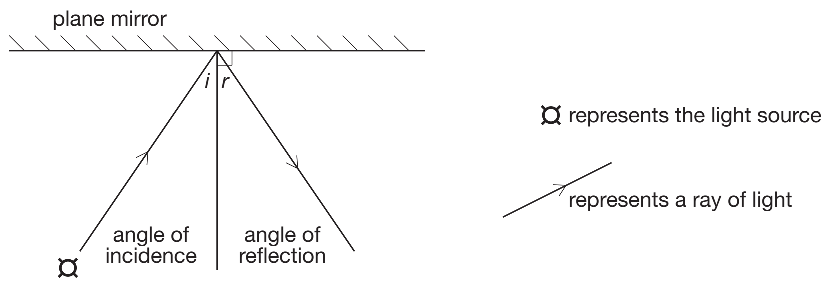
Be prepared to explain your reasoning.

Repeat for regular hexagrams and octograms.





Applying properties of parallel lines to problems with plane mirrors

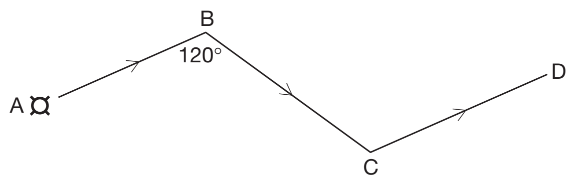


You may already know from your work in science that the angle of incidence (i) is equal to the angle of reflection (r).

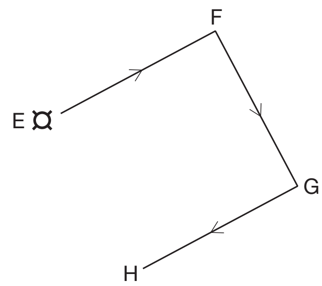
Take this as a given in the following problems.

Given: $i = r$

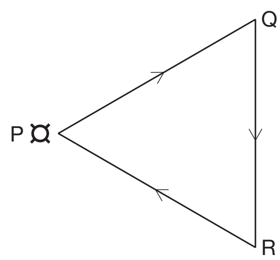
1 Position two plane mirrors so that AB is parallel to CD.



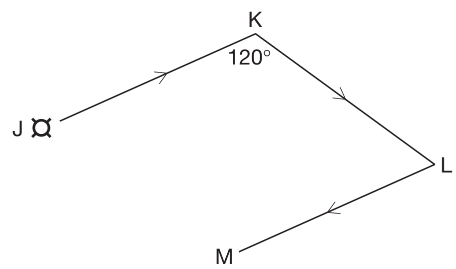
2 Position two plane mirrors so that ∠EFG = ∠FGH = 90°.



3 Position two plane mirrors so that triangle PQR is equilateral.



4 Position two plane mirrors so that JK is parallel to LM.



What is the angle between the two mirrors in each question? Are there any general rules for placing the mirrors so that the last light ray is parallel to the first?

